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Hierarchy of Chaotic Maps with an Invariant Measure

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We give hierarchy of one-parameter family $\Phi(\alpha, x)$ of maps at the interval $[0, 1]$ with an invariant measure. Using the measure, we calculate Kolmogorov-Sinai entropy, or equivalently Lyapunov characteristic exponent of these maps analytically, where the results thus obtained have been approved with the numerical simulation. In contrary to the usual one-parameter family of maps such as logistic and tent maps, these maps do not possess period doubling or period-n-tupling cascade bifurcation to chaos, but they have single fixed point attractor for certain values of the parameter, where they bifurcate directly to chaos without having period-n-tupling scenario exactly at those values of the parameter whose Lyapunov characteristic exponent begins to be positive.

KEY WORDS: Chaos; invariant measure; entropy; Lyapunov characteristic exponent; ergodic dynamical systems.

1. INTRODUCTION

In recent years chaos or more properly dynamical systems have become an important area of research activity. One of the landmarks in it was the introduction of the concept of Sinai–Ruelle–Bowen (SRB) measure or natural invariant measure. This is roughly speaking a measure that is supported on an attractor and it describes the statistics of the long time

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behavior of the orbits for almost every initial condition in the corresponding basin of attractor. This measure can be obtained by computing the fixed points of the so called Frobenius–Perron (FP) operator which can be viewed as a differential-integral operator, hence, exact determination of invariant measure of dynamical systems is rather a nontrivial task, such that invariant measure of few dynamical systems such as one-parameter family one-dimensional piecewise linear maps[1–3] including Baker and tent maps or unimodal maps such as logistic map for certain values of its parameter, can be derived analytically. In most of cases only numerical algorithms, as an example Ulam's method[4–6] are used for computation of fixed points of FP operator. Here in this article we give a hierarchy of one-parameter family $\Phi^{(1,2)}(\alpha, x)$ of maps of interval $[0, 1]$ with an invariant measure. These maps are defined as ratios of polynomials of degree N, where we have derived analytically their invariant measure for an arbitrary values of the parameter α and every integer values of N. Using this measure, we have calculated analytically, Kolmogorov–Sinai (KS) or equivalently Lyapunov characteristic exponent of these maps, where the numerical simulation of up to degree $N=10$ confirm the analytic calculations. Also it is shown that these maps have another interesting property, that is, for even values of N the $\Phi^{(1)}(\alpha, x)(\Phi^{(2)}(\alpha, x))$ maps have only a fixed point attractor $x=1(x=0)$ provided that their parameter belongs to interval $(N, \infty)((0, \frac{1}{N}))$ while, at $\alpha \geqslant N$ $(\alpha \geqslant \frac{1}{N})$ they bifurcate to chaotic regime without having any period doubling or period-n-tupling scenario and remain chaotic for all $\alpha \in (0, N)$ $(\alpha \in (\frac{1}{N}, \infty))$ but for odd values of N, these maps have only fixed point attractor $x=0$ for $\alpha \in (\frac{1}{N}, N)$, again they bifurcate to a chaotic regime at $\alpha \geqslant \frac{1}{N}$, and remain chaotic for $\alpha \in (0, \frac{1}{N})$, finally they bifurcate at $\alpha = N$ to have $x=1$ as fixed point attractor for all $\alpha \in (\frac{1}{N}, \infty)$. The paper is organized as follows: in Section 2 we introduce the hierarchy of family of one-parameter maps, In Section 3 we show that the proposed anzatz for the invariant measure of their maps are eigenfuntion of FP operator with largest eigenvalue 1, for any finite N. Then in Section 4 using this measure, we calculate KS-entropy of these maps for an arbitrary values of the parameter α and every integer values of N. In Section 5 we compare the analytic calculations with the numerical simulations. The paper ends with a brief conclusion.

2. ONE-PARAMETER FAMILIES OF CHAOTIC MAPS

The one-parameter families of chaotic maps of the interval $[0, 1]$ with an invariant measure are defined as the ratio of polynomials of degree N:

$$\Phi_N^{(1)}(x,\alpha)=\frac{\alpha^2(1+(-1)^N\,{}_2F_1(-N,N,\frac{1}{2},x))}{(\alpha^2+1)+(\alpha^2-1)(-1)^N\,{}_2F_1(-N,N,\frac{1}{2},x)}$$

$$=\frac{\alpha^2(T_N(\sqrt{x}))^2}{1+(\alpha^2-1)(T_N(\sqrt{x})^2)},\tag{2.1}$$

$$\Phi_N^{(2)}(x,\alpha)=\frac{\alpha^2(1-(-1)^N\,{}_2F_1(-N,N,\frac{1}{2},(1-x)))}{(\alpha^2+1)-(\alpha^2-1)(-1)^N\,{}_2F_1(-N,N,\frac{1}{2},(1-x))}$$

$$=\frac{\alpha^2(U_N(\sqrt{(1-x)}))^2}{1+(\alpha^2-1)(U_N(\sqrt{(1-x)})^2)},\tag{2.2}$$

where N is an integer greater than one. Also

$${}_2F_1(-N,N,\tfrac{1}{2},x)=(-1)^N\cos(2N\text{ arc cos }\sqrt{x})=(-1)^N\,T_{2N}(\sqrt{x})$$

is the hypergeometric polynomials of degree N and $T_N(U_n(x))$ are Chebyshev polynomials of type I (type II), respectively. Obviously these map the unit interval $[0,1]$ into itself and are related to each other through the following relation:

$$\Phi_N^{(1)}(x,\alpha)=g\left(\Phi_N^{(2)},g(x),\frac{1}{\alpha}\right)=g\circ\Phi_N^{(1)}\left(\frac{1}{\alpha}\right)\circ g(x)\qquad\text{for even N}\tag{2.3}$$

and

$$\Phi_N^{(1)}(x,\alpha)=\Phi_N^{(2)}(x,\alpha)\qquad\text{for odd N},\tag{2.4}$$

where $g(x)$ is the invertible map $g(x)=g^{-1}(x)=1-x$ and the symbol $\circ$ means the composition of functions. From now on, depending on the situation we will consider one of these maps, since, we can get all required information concerning the other map via using the relations (2.3) and (2.4) between these two maps. $\Phi_N^{(1)}(\alpha,x)$ is (N-1)-model map, that is it has $(N-1)$ critical points in unit interval $[0,1]$, since its derivative is proportional to derivative of hypergeometric polynomial ${}_2F_1(-N,N,\frac{1}{2},x)$ which is itself a hypergeometric polynomial of degree $(N-1)$, hence it has $(N-1)$ real roots in unit interval $[0,1]$. Defining Shwarzian derivative $S\Phi_N(x)$ as:

$$S(\Phi_N^{(1)}(x))=\frac{\Phi_N^{(1)'''}(x)}{\Phi_N^{(1)'}(x)}-\frac{3}{2}\left(\frac{\Phi_N^{(1)''}(x)}{\Phi_N^{(1)'}(x)}\right)^2=\left(\frac{\Phi_N^{(1)''}(x)}{\Phi_N^{(1)'}(x)}\right)'-\frac{1}{2}\left(\frac{\Phi_N^{(1)''}(x)}{\Phi_N^{(1)'}(x)}\right)^2,$$

with a prime denoting a single differential, one can show that:

$$S(\Phi_N^{(1)}(x))=S({}_2F_1(-N,N,\tfrac{1}{2},x))\leqslant 0,$$

since $\frac{d}{dx}\left({}_2F_1(-N, N, \frac{1}{2}, x)\right)$ can be written as:

$$\frac{d}{dx}\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right) = A\prod_{i=1}^{N-1}(x-x_i),$$

with $0 \leqslant x_1 < x_2 < x_3 < \cdots < x_{N-1} \leqslant 1$, then we have:

$$S\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right) = \frac{-1}{2}\sum_{J=1}^{N-1}\frac{1}{(x-x_j)^2} - \left(\sum_{J=1}^{N-1}\frac{1}{(x-x_j)}\right)^2 < 0.$$

Therefore, the maps $\Phi_N^{(1)}(x)$ have at most $N+1$ attracting periodic orbits.[7] As we will show at the end of this section, these maps have only a single period one stable fixed points.

Denoting n-composition of functions $\Phi^{(1,2)}(x_1, \alpha)$ by $\Phi^{(n)}$, it is straightforward to show that the derivative of $\Phi^{(n)}$ at its possible n periodic points of an n-cycle: $x_2 = \Phi^{(1,2)}(x_1, \alpha)$, $x_3 = \Phi^{(1,2)}(x_2, \alpha)$, ..., $x_1 = \Phi^{(1,2)}(x_n, \alpha)$ is

$$\left|\frac{d}{dx}\Phi^{(n)}\right| = \left|\frac{d}{dx}\overbrace{(\Phi^{(1,2)}\circ\Phi^{(1,2)}\circ\cdots\circ\Phi^{(1,2)}(x,\alpha))}^{n}\right| = \prod_{k=1}^{n}\left|\frac{N}{\alpha}(\alpha^2+(1-\alpha^2)\,x_k)\right|, \tag{2.5}$$

since for $x_k \in [0, 1]$ we have:

$$min(\alpha^2+(1-\alpha^2 x_k)) = min(1, \alpha^2),$$

therefore,

$$min\left|\frac{d}{dx}\Phi^{(n)}\right| = \left(\frac{N}{\alpha}min(1, \alpha^2)\right)^n.$$

Hence the above expression is definitely greater than one for $\frac{1}{N} < \alpha < N$, that is: both maps do not have any kind of n-cycle or periodic orbits for $\frac{1}{N} < \alpha < N$, actually they are ergodic for this interval of parameter. From (2.5) it follows the values of $|\frac{d}{dx}\Phi^{(n)}|$ at n periodic points of the n-cycle belongs to interval [0,1], varies between $(N\alpha)^n$ and $(\frac{N}{\alpha})^n$ for $\alpha < \frac{1}{N}$ and between $(\frac{N}{\alpha})^n$ and $(N\alpha)^n$ for $\alpha > N$, respectively.

From the definition of these maps, we see that for odd N, both $x = 0$ and $x = 1$ belong to one of the n-cycles, while for even N, only $x = 1$ belongs to one of the n-cycles of $\Phi_N^{(1)}(x, \alpha)$ and $x = 0$ belongs to one of the n-cycles of $\Phi_N^{(2)}(x, \alpha)$.

For $\alpha < (\frac{1}{N})(\alpha > N)$, the formula (2.5) implies that for those cases in which $x = 0$ $(x = 1)$ belongs to one of n-cycles we will have $|\frac{d}{dx}\Phi^{(n)}| < 1$, hence the curve of $\Phi^{(n)}$ starts at $x = 0(x = 1)$ beneath the bisector and then crosses it at the next (previous) periodic point with slope greater than one, since the formula (2.5) implies that the slope of fixed points increases with the increasing (decreasing) of $|x_k|$, therefore at all periodic points of n-cycles except for $x=0(x=1)$ the slope is greater than one that is they are unstable, this is possible only if $x=0(x=1)$ is the only period one fixed point of these maps.

Hence all n-cycles except for possible period one fixed points $x=0$ and $x=1$ are unstable, where for $\alpha \in [0, \frac{1}{N}]$, the fixed point $x=0$ is stable in maps $\Phi_N^{(1,2)}(x, \alpha)$ (for odd N) and $\Phi_N^{(2)}(x, \alpha)$ (for even N), while for $\alpha \in [N, \infty)$ and $\Phi_N^{(1)}(x, \alpha)$, the $x=1$ is stable fixed point in maps $\Phi_N^{(1,2)}(x, \alpha)$ (for odd N).

As an example we give below some of these maps:

$$\phi_2^{(1)} = \frac{\alpha^2(2x-1)^2}{4x(1-x)+\alpha^2(2x-1)^2}, \qquad \phi_2^{(2)} = \frac{4\alpha^2 x(1-x)}{1+4(\alpha^2-1)\, x(1-x)},$$

$$\phi_3^{(1)} = \phi_3^{(2)} = \frac{\alpha^2 x(4x-3)^2}{\alpha^2 x(4x-3)^2+(1-x)(4x-1)^2},$$

$$\phi_4^{(1)} = \frac{\alpha^2(1-8x(1-x))^2}{\alpha^2(1-8x(1-x))^2+16x(1-x)(1-2x)^2},$$

$$\phi_4^{(2)} = \frac{16\alpha^2 x(1-x)(1-2x)^2}{(1-8x+8x^2)^2+16\alpha^2 x(1-x)(1-2x)^2},$$

$$\phi_5^{(1)} = \phi_5^{(2)} = \frac{\alpha^2 x(16x^2-20x+5)^2}{\alpha^2 x(16x^2-20x+5)^2+(1-x)(16x^2-(2x-1))}.$$

Below we also introduce their conjugate or isomorphic maps which will be very useful in derivation of their invariant measure and calculation of their KS-entropy in the next sections. Conjugacy means that the invertible map $h(x)=\frac{1-x}{x}$ maps $I=[0,1]$ into $[0,\infty)$ and transform maps $\Phi_N^{(1,2)}(x, \alpha)$ into $\tilde{\Phi}_N^{(1,2)}(x, \alpha)$ defined as:

$$\begin{cases} \tilde{\Phi}_N^{(1)}(x, \alpha) = h \circ \Phi_N^{(1)}(x, \alpha) \circ h^{(-1)} = \dfrac{1}{\alpha^2} \tan^2(N \arctan \sqrt{x}), \\ \tilde{\Phi}_N^{(2)}(x, \alpha) = h \circ \Phi_N^{(2)}(x, \alpha) \circ h^{-1} = \dfrac{1}{\alpha^2} \cot^2\left(N \arctan \dfrac{1}{\sqrt{x}}\right). \end{cases} \tag{2.6}$$

3. INVARIANT MEASURE

Dynamical systems, even apparently simple dynamical systems which are described by maps of an interval can display a rich variety of different asymptotic behavior. On measure theoretical level these type of behavior are described by SRB[8] or invariant measure describing statistically stationary states of the system. The probability measure μ on $[0, 1]$ is called an SRB or invariant measure of the maps $y = \Phi_N^{(1,2)}(x, \alpha)$ given in (2.1) and (2.2), if it is $\Phi_N^{(1,2)}(x, \alpha)$-invariant and absolutely continuous with respect to Lebesgue measure. For deterministic system such as $\Phi_N^{(1,2)}(x, \alpha)$-map, the $\Phi_N^{(1,2)}(x, \alpha)$-invariance means that, its invariant measure $\mu(x)$ fulfills the following formal FP integral equation

$$\mu(y) = \int_0^1 \delta(y - \Phi_N^{(1,2)}(x, \alpha))\, \mu(x)\, dx.$$

This is equivalent to:

$$\mu(y) = \sum_{x \in \Phi_N^{-1(1,2)}(y, \alpha)} \mu(x) \frac{dx}{dy}, \tag{3.1}$$

defining the action of standard FP operator for the map $\Phi_N(x)$ over a function as:

$$P_{\Phi_N^{(1,2)}} f(y) = \sum_{x \in \Phi_N^{-1(1,2)}(y, \alpha)} f(x) \frac{dx}{dy}. \tag{3.2}$$

We see that, the invariant measure $\mu(x)$ is actually the eigenstate of the FP operator $P_{\Phi_N^{(1,2)}}$ corresponding to largest eigenvalue 1.

As we will prove below the measure $\mu_{\Phi_N^{(1,2)}(x,\alpha)}(x, \beta)$ defined as:

$$\frac{1}{\pi} \frac{\sqrt{\beta}}{\sqrt{x(1-x)}\, (\beta + (1-\beta)\, x)} \tag{3.3}$$

with $\beta > 0$ is the invariant measure of the maps $\Phi_N^{(1,2)}(x, \alpha)$ provided that, we choose the parameter α in the following form:

$$\alpha = \frac{\sum_{k=0}^{[\frac{(N-1)}{2}]} C_{2k+1}^N \beta^{-k}}{\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^{-k}} \tag{3.4}$$

in $\Phi_N^{(1,2)}(x, \alpha)$ maps for odd values of N and

$$\alpha = \frac{\beta \sum_{k=0}^{[\frac{(N)}{2}]} C_{2k}^N \beta^{-k}}{\sum_{k=0}^{[\frac{(N-1)}{2}]} C_{2k+1}^N \beta^{-k}} \tag{3.5}$$

in $\Phi_N^{(2)}(x, \alpha)$ maps for even values of N, where the symbol [ ] means greatest integer part.

As we see the above measure is defined only for $\beta > 0$, hence from the relations (3.4) and (3.5), it follows that the maps $\Phi_N^{(1,2)}(x, \alpha)$ have invariant measure only for $\alpha \in (\frac{1}{N}, N)$ for odd value of N, $\Phi_N^{(1)}(x, \alpha)$ maps have invariant measure for $\alpha \in (0, N)$ and $\Phi_N^{(2)}(x, \alpha)$ have for $\alpha \in (\frac{1}{N}, \infty)$ for even N, respectively. For other values of α these maps have single attractive fixed points, which is the same as the prediction of the previous section.

In order to prove that the measure (3.3) satisfies equation (3.1), with α given in (3.4) and (3.5), it is rather convenient to consider the conjugate map:

$$\tilde{\Phi}_N^{(1)}(x, \alpha) = \frac{1}{\alpha^2} \tan^2(N \operatorname{arc\,tan} \sqrt{x}) \tag{3.6}$$

with measure $\tilde{\mu}_{\tilde{\Phi}_N^{(1)}}$ related to the measure $\mu_{\tilde{\Phi}_N^{(1)}}$ to the following relation:

$$\tilde{\mu}_{\tilde{\Phi}_N^{(1)}}(x) = \frac{1}{(1+x)^2}\, \mu_{\Phi_N^{(1)}}\left(\frac{1}{1+x}\right).$$

Denoting $\tilde{\Phi}_N^{(1)}(x, \alpha)$ on the left hand side of (3.6) by y and inverting it, we get:

$$x_k = \tan^2\left(\frac{1}{N} \operatorname{arc\,tan} \sqrt{y\alpha^2} + \frac{k\pi}{N}\right) \qquad k = 1, ..., N. \tag{3.7}$$

Then, taking derivative of x_k with respect to y, we obtain:

$$\left|\frac{dx_k}{dy}\right| = \frac{\alpha}{N} \sqrt{x_k}\,(1+x_k)\, \frac{1}{\sqrt{y}\,(1+\alpha^2 y)}. \tag{3.8}$$

Substituting the above result in equation (3.1), we have:

$$\tilde{\mu}_{\tilde{\Phi}_N^{(1)}}(y) \sqrt{y}\,(1+\alpha^2 y) = \frac{\alpha}{N} \sum_k \sqrt{x_k}\,(1+x_k)\, \tilde{\mu}_{\tilde{\Phi}_N^{(1)}}(x_k), \tag{3.9}$$

considering the following anatz for the invariant measure $\tilde{\mu}_{\tilde{\Phi}_N^{(1)}}(y)$:

$$\tilde{\mu}_{\tilde{\Phi}_N^{(1)}}(y) = \frac{\sqrt{\beta}}{\sqrt{y}\,(1+\beta y)}, \tag{3.10}$$

the above equation reduces to:

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{\alpha}{N} \sum_{k=1}^{N} \left(\frac{1+x_k}{1+\beta x_k}\right)$$

which can be written as:

$$\frac{1+\alpha^2 y}{1+\beta y}=\frac{\alpha}{\beta}+\left(\frac{\beta-1}{\beta^2}\right)\frac{\partial}{\partial\beta^{-1}}(\ln(\Pi_{k=1}^{N}(\beta^{-1}+x_k))). \tag{3.11}$$

To evaluate the second term in the right hand side of above formulas we can write the equation in the following form:

$$\begin{aligned}0&=\alpha^2 y\cos^2(N\arctan\sqrt{x})-\sin^2(N\arctan\sqrt{x})\\&=\frac{(-1)^N}{(1+x)^N}\left(\alpha^2 y\left(\sum_{k=0}^{[\frac{N}{2}]}C_{2k}^{N}(-1)^N x^k\right)^2-x\left(\sum_{k=0}^{[\frac{N-1}{2}]}C_{2k+1}^{N}(-1)^N x^k\right)^2\right),\\&=\frac{\text{constant}}{(1+x)^N}\prod_{k=1}^{N}(x-x_k),\end{aligned}$$

where x_k are the roots of equation (3.6) and they are given by formula (3.7). Therefore, we have:

$$\begin{aligned}&\frac{\partial}{\partial\beta^{-1}}\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k)\right)\\&\quad=\frac{\partial}{\partial\beta^{-1}}\ln[(1-\beta^{-1})^N(\alpha^2 y\cos^2(N\arctan\sqrt{-\beta^{-1}})\\&\quad-\sin^2(N\arctan\sqrt{-\beta^{-1}}))]\\&\quad=-\frac{N\beta}{\beta-1}+\frac{\beta N(1+\alpha^2 y)\,A\left(\frac{1}{\beta}\right)}{\left(A\left(\frac{1}{\beta}\right)\right)^2\beta^2 y+\left(B\left(\frac{1}{\beta}\right)\right)^2},\end{aligned} \tag{3.12}$$

with polynomials $A(x)$ and $B(x)$ defined as:

$$\begin{aligned}A(x)&=\sum_{k=0}^{[\frac{N}{2}]}C_{2k}^{N}x^k,\\B(x)&=\sum_{k=0}^{[\frac{N-1}{2}]}C_{2k+1}^{N}x^k.\end{aligned} \tag{3.13}$$

In deriving of above formulas we have used the following identities:

$$\begin{aligned}\cos(N\arctan\sqrt{x})&=\frac{A(-x)}{(1+x)^{\frac{N}{2}}},\\\sin(N\arctan\sqrt{x})&=\sqrt{x}\,\frac{B(-x)}{(1+x)^{\frac{N}{2}}},\end{aligned} \tag{3.14}$$

inserting the results (3.12) in (3.11), we get:

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{1+\alpha^2 y}{\left(\frac{B(\frac{1}{\beta})}{\alpha A(\frac{1}{\beta})} + \beta\left(\frac{\alpha A(\frac{1}{\beta})}{B(\frac{1}{\beta})}\right) y\right)}.$$

Hence to get the final result we have to choose the parameter α as:

$$\alpha = \frac{B\left(\frac{1}{\beta}\right)}{A\left(\frac{1}{\beta}\right)}.$$

With a procedure similar to the one given above we could get the relation (3.5) between the parameters α and β for the second kind of maps.

4. KOLMOGROV–SINAI ENTROPY

KS-entropy or metric entropy measures how chaotic a dynamical system is and it is proportional to the rate at which information about the state of dynamical system is lost in the course of time or iteration. Therefore, it can also be defined as the average rate of loss of information for a discrete measurable dynamical system $(\Phi_N^{(1,2)}(x,\alpha), \mu)$. By introducing a partition $\alpha = A_c(n_1, \ldots\ldots n_\gamma)$ of the interval $[0, 1]$ into individual laps A_i, one can define the usual entropy associated with the partition by:

$$H(\mu, \gamma) = -\sum_{i=1}^{n(\gamma)} m(A_c) \ln m(A_c),$$

where $m(A_c) = \int_{n \in A_i} \mu(x)\, dx$ is the invariant measure of A_i. Defining a n-th refining $\gamma(n)$ of γ:

$$\gamma^n = \bigcup_{k=0}^{n-1} (\Phi_N^{(1,2)}(x,\alpha))^{-(k)}(\gamma)$$

then defining an entropy per unit step of refining is defined by:

$$h(\mu, \Phi_N^{(1,2)}(x,\alpha), \gamma) = \lim_{n\to\infty}\left(\frac{1}{n} H(\mu, \gamma)\right),$$

now, if the size of individual laps of $\gamma(N)$ tends to zero as n increases, the above entropy reduces to well known as KS-entropy, that is:

$$h(\mu, \Phi_N^{(1,2)}(x,\alpha)) = h(\mu, \Phi_N^{(1,2)}(x,\alpha), \gamma).$$

KS-entropy is actually a quantitative measure of the rate of information lost with the refining and it can be written as:

$$h(\mu, \Phi_N^{(1,2)}(x,\alpha)) = \int \mu(x)\, dx \ln \left| \frac{d}{dx} \Phi_N^{(1,2)}(x,\alpha) \right|, \tag{4.1}$$

which is also a statistical mechanical expression for the Lyapunov Characteristic exponent, that is: mean divergence rate of two nearby orbits. The measurable dynamical system $(\Phi_N^{(1,2)}(x,\alpha), \mu)$ is chaotic for $h>0$ and predictive for $h=0$.

In order to calculate the KS-entropy of the maps $\Phi_N^{(1,2)}(x,\alpha)$, it is rather convenient to consider their conjugate maps given in (2.6), since it can be shown that KS-entropy is a kind of differentiable invariant, that is, it is preserved under the conjugacy map, hence we have:

$$h(\mu, \Phi_N^{(1,2)}(x,\alpha)) = h(\tilde{\mu}, \tilde{\Phi}_N^{(1,2)}(x,\alpha)). \tag{4.2}$$

Using the integral (4.1), the KS-entropy of $\Phi_N^{(2)}(\alpha, x)$ can be written as

$$\begin{aligned} h(\mu, \Phi_N^{(2)}(x,\alpha)) &= h(\tilde{\mu}, \tilde{\Phi}_N^{(2)}(x,\alpha)) \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\, dx}{\sqrt{x}\,(1+\beta x)} \ln \left| \frac{1}{\alpha^2}\frac{d}{dx}(\cot^2(N \arctan\sqrt{x})) \right| \end{aligned} \tag{4.3}$$

Using the relations given in (3.14) we get

$$h(\mu, \Phi_N^{(2)}(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\, dx}{\sqrt{x}\,(1+\beta x)} \ln \left(\frac{N}{\alpha^2} \left| \frac{(1+x)^{N-1} A(-x)}{x^2 (B(-x))^3} \right| \right), \tag{4.4}$$

for even N, and

$$h(\mu, \Phi_N^{(2)}(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\, dx}{\sqrt{x}\,(1+\beta x)} \ln \left(\frac{N}{\alpha^2} \left| \frac{(1+x)^{N-1} B(-x)}{(A(-x))^3} \right| \right). \tag{4.5}$$

for odd N.

Considering again the relations given in (3.14), we see that polynomials appearing in the numerator (denominator) of integrand appearing on the right hand side of equation $(4-5)$, have $\frac{[N-1]}{2}$ $(\frac{[N]}{2})$ simple roots, denoted by $x_k^B \quad k=1,\ldots,[\frac{N-1}{2}]$ $(x_k^A \quad k=1,\ldots,[\frac{N}{2}])$ in the interval $[0,\infty)$. Hence, we can write the above formula in the following form:

$$h(\mu, \Phi_N^{(2)}(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\, dx}{\sqrt{x}\,(1+\beta x)} \ln \left(\frac{N}{\alpha^2} \times \frac{(1+x)^{N-1} \prod_{k=1}^{[\frac{N}{2}]} |x - x_k^A|}{x^2 \prod_{k=1}^{[\frac{N-1}{2}]} |x - x_k^B|^3} \right),$$

for even N, and

$$h(\mu, \Phi_N^{(2)}(x, \alpha)) = \frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{\beta}\, dx}{\sqrt{x}\,(1+\beta x)} \ln\left(\frac{N}{\alpha^2} \times \frac{(1+x)^{N-1} \prod_{k=1}^{[\frac{N-1}{2}]} |x - x_k^B|}{\prod_{k=1}^{[\frac{N}{2}]} |x - x_k^A|^{3}}\right).$$

for odd N.

Now making the following change of variable $x = \frac{1}{\beta}\tan^2\frac{\Theta}{2}$, and taking into account that degree of numerators and denominators are equal for both even and odd values of N, hence we get

$$h(\alpha, \Phi_N^{(2)}(x, \alpha)) = \frac{1}{\pi}\int_0^{\infty} d\theta \left\{ \ln\left(\frac{N}{\alpha^2}\right) + (N-1)\ln|\beta + 1 + (\beta - 1)\cos\theta| \right.$$
$$+ \sum_{k=1}^{[\frac{N}{2}]} \ln|1 - x_k^A\beta + (1 + x_k^A\beta)\cos\theta|$$
$$- 3\sum_{k=1}^{[\frac{N-1}{2}]} \ln|1 - x_k^B\beta + (1 + x_k^B\beta)\cos\theta|$$
$$\left. - 2\ln|1 + \cos\theta| \right\},$$

for even and

$$h(\alpha, \Phi_N^{(2)}(x, \alpha)) = \frac{1}{\pi}\int_0^{\infty} d\theta \left\{ \ln\left(\frac{N}{\alpha^2}\right) + (N-1)\ln|\beta + 1 + (\beta - 1)\cos\theta| \right.$$
$$+ \sum_{k=1}^{[\frac{N-1}{2}]} \ln|1 - x_k^B\beta + (1 + x_k^B\beta)\cos\theta|$$
$$\left. - 3\sum_{k=1}^{[\frac{N}{2}]} \ln|1 - x_k^A\beta + (1 + x_k^A\beta)\cos\theta| \right\}.$$

for odd N.

Using the following integrals:

$$\frac{1}{\pi}\int_0^{\pi} \ln|a + b\cos\theta| = \begin{cases} \ln\left|\dfrac{a + \sqrt{a^2 - b^2}}{2}\right| & |a| > |b| \\ \ln\left|\dfrac{b}{2}\right| & |a| \leqslant |b|, \end{cases}$$

we get

$$h(\alpha, \Phi_N^{(2)}(x,\alpha)) = \begin{cases} \ln\left(\dfrac{N}{\alpha^2}\dfrac{(\beta+1+2\sqrt{\beta})^{N-1}\prod_{k=1}^{[\frac{N}{2}]}(1+x_k^A\beta)}{(\prod_{k=1}^{[\frac{N-1}{2}]}(1+x_k^B\beta))^3}\right) & \text{for even N} \\ \ln\left(\dfrac{N}{\alpha^2}\dfrac{(\beta+1+2\sqrt{\beta})^{N-1}\prod_{k=1}^{[\frac{N-1}{2}]}(1+x_k^B\beta)}{(\prod_{k=1}^{[\frac{N}{2}]}(1+x_k^A\beta))^3}\right) & \text{for odd N,} \end{cases}$$

or

$$h(\alpha, \Phi_N^{(2)}(x,\alpha)) = \begin{cases} \ln\left(\dfrac{N}{\alpha^2}\dfrac{(1+\beta+2\sqrt{\beta})^{N-1}\beta^2 A(\frac{1}{\beta})}{\beta^{(N-1)}(B(\frac{1}{\beta}))^3}\right) & \text{for even N} \\ \ln\left(\dfrac{N}{\alpha^2}\dfrac{(1+\beta+2\sqrt{\beta})^{N-1} B(\frac{1}{\beta})}{\beta^{(N-1)}(A(\frac{1}{\beta}))^3}\right) & \text{for odd N} \end{cases}$$

Finally using the relation:

$$\alpha = \begin{cases} \beta\dfrac{A(\frac{1}{\beta})}{B(\frac{1}{\beta})} & \text{for even N} \\ \dfrac{B(\frac{1}{\beta})}{A(\frac{1}{\beta})} & \text{for odd N,} \end{cases}$$

we get

$$h(\mu, \Phi_N^{(2)}(x,\alpha)) = \ln\left(\frac{N(1+\beta+2\sqrt{\beta})^{N-1}}{(\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N\beta^k)(\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N\beta^k)}\right). \tag{4.6}$$

With a calculation rather similar to the one given above, we can calculate the KS-entropy of the $\Phi_N^1(x,\alpha)$ maps where the results are the same as those given by (4.6).

The KS-entropy (4.6) is invariant with respect to $\beta \longrightarrow (\frac{1}{\beta})$, therefore, it has the same asymptotic behavior near $\beta \longrightarrow 0$ and $\beta \longrightarrow \infty$, where its asymptotic forms read

$$\begin{cases} h(\mu, \Phi_N^{(1,2)}(x, \alpha = N+0^-)) \sim (N-\alpha)^{\frac{1}{2}} \\ h(\mu, \Phi_N^{(1,2)}(x, \alpha = \frac{1}{N}+0^+)) \sim (\alpha-\frac{1}{N})^{\frac{1}{2}}, \end{cases}$$

for odd N and:

$$\begin{cases} h(\mu, \Phi_N^{(1)}(x, \alpha = N + 0^-)) \sim (N - \alpha)^{\frac{1}{2}} \\ h(\mu, \Phi_N^{(2)}(x, \alpha = \frac{1}{N} + 0^+)) \sim (\alpha - \frac{1}{N})^{\frac{1}{2}}. \end{cases}$$

for even N. The above asymptotic form indicates that the maps, $\Phi_N^{(1,2)}(x, \alpha)$ belong to the same universality class which are different from the universality class of pitch fork bifurcating maps but their asymptotic behavior is similar to class of intermittent maps,[10] even though intermittency can not occur in these maps for any values of parameter α, since the maps $\Phi_N^{(1,2)}(x, \alpha)$ and their n-composition $\Phi^{(n)}$ do not have minimum values other than zero and maximum values other than one in the interval $[0, 1]$.

5. SIMULATION

Here in this section we try to calculate Lyapunov characteristic exponent of maps $\Phi_N^{(1,2)}(x, \alpha)$, $N = 1, 2, \ldots, 5$, in order to investigate these maps numerically. In fact, Lyapunov characteristic exponent is the characteristic

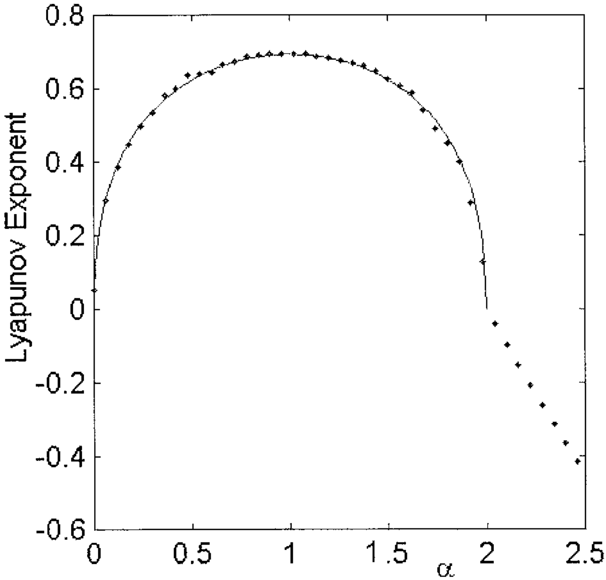

Fig. 1. Solid curve shows the variation of KS-entropy of $\Phi_2^{(1)}(x, \alpha)$, in terms of the parameter α dotted curve shows the variation of Lyapunov characteristic exponent of $\Phi_2^{(1)}(x, \alpha)$, in terms of the parameter α.

exponent of the rate of average magnification of the neighborhood of an arbitrary point X_0 and it is denoted by $\Lambda(x_0)$ which is written as:

$$\Lambda^{(1,2)}(x_0) = \lim_{n\to\infty} \ln \mid \overbrace{\Phi_N^{(1,2)}(x,\alpha) \circ \Phi_N^{(1,2)} \cdots \circ \Phi_N^{(1,2)}(x_K,\alpha)}^{n} \mid$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \ln \left| \frac{d\Phi_N^{(1,2)}(x_k,\alpha)}{dx} \right|, \tag{5.1}$$

where $x_k = \overbrace{\Phi_N^{(1,2)} \circ \Phi_N^{(1,2)} \circ \ldots. \circ \Phi_N^{(1,2)}(x_0)}^{k}$. It is obvious that $\Lambda^{(1,2)}(x_0) < 0$ for an attractor, $\Lambda^{(1,2)}(x_0) > 0$ for a repeller and $\Lambda^{(1,2)}(x_0) = 0$ for marginal situation. Also the Lyapunov number is independent of initial point x_0, provided that the motion inside the invariant manifold is ergodic, thus $\Lambda^{(1,2)}(x_0)$ characterizes the invariant manifold of $\Phi_N^{(1,2)}$ as a whole. For the values of parameter α, such that the map $\Phi_N^{(1,2)}$ be measurable, Birkhoff ergodic theorem implies the equality of KS-entropy and Lyapunov number, that is:

$$h(\mu, \Phi_n^{(1,2)}) = \Lambda^{(1,2)}(x_0, \Phi_N^{(1,2)}), \tag{5.2}$$

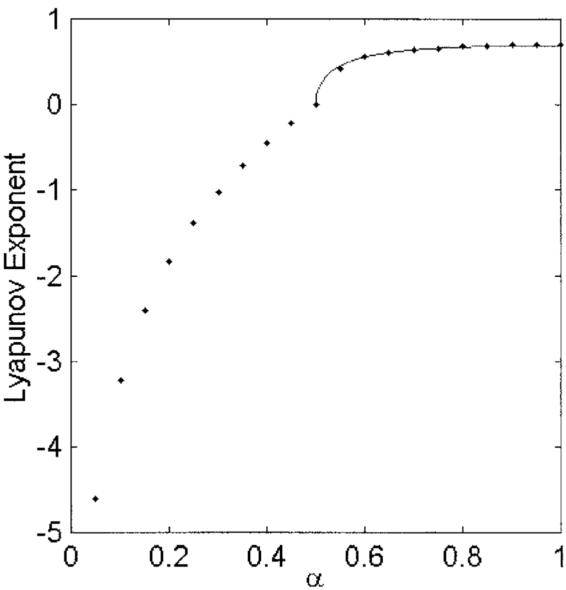

Fig. 2. Solid curve shows the variation of KS-entropy of $\Phi_2^{(2)}(x,\alpha)$, in terms of the parameter α dotted curve shows the variation of Lyapunov characteristic exponent of $\Phi_2^{(2)}(x,\alpha)$, in terms of the parameter α.

Comparison of analytically calculated KS-entropy of maps $\Phi_N^{(1,2)}(x, \alpha)$ for $N = 1, 2, \ldots 10$, (see Figures 1, 2 and 3 for $N = 2$ and 3) with the corresponding Lyapunov characteristic exponent obtained by the simulation, indicates that in chaotic region, these maps are ergodic as Birkohf ergodic theorem predicts. In the non-chaotic region of the parameter, Lyapunov characteristic exponent is negative definite, since in this region we have only single period fixed points without bifurcation. Therefore, combining the analytic discussion of section 2 with the numerical simulation we deduce that: these maps are ergodic in certain values of their parameter as explained above and in complementary interval of parameter they have only a single period one attractive fixed point, such that in contrary to the most of usual one-dimensional one-parameter family of maps they have only bifurcation from a period one attractive fixed point to chaotic state or vise versa.

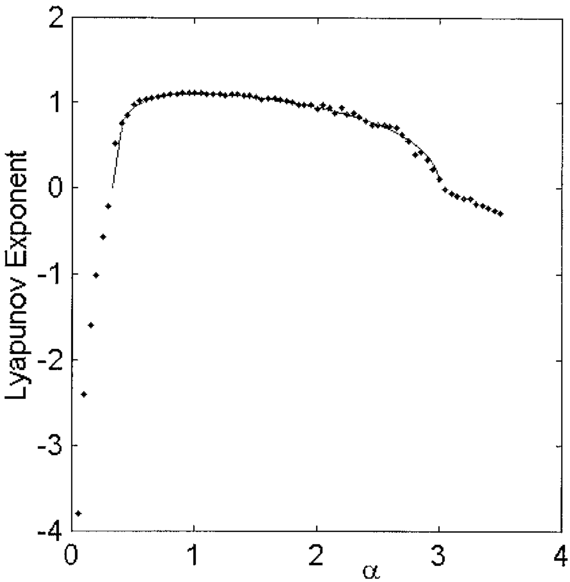

Fig. 3. Solid curve shows the variation of KS-entropy of $\Phi_3^{(1)}(x, \alpha)$, in terms of the parameter α, dotted curve shows variation of Lyapunov characteristic exponent of $\Phi_3^{(1)}(x, \alpha)$, in terms of the parameter α.

6. CONCLUSION

We have given hierarchy of exactly solvable one-parameter family of one-dimensional chaotic maps with an invariant measure, that is measurable dynamical system with an interesting property of being either chaotic (It is more appropriate to say ergodic) or having stable period one fixed point and they bifurcate from a stable single periodic state to chaotic one and vice-versa without having usual period doubling or period-n-tupling scenario.

Perhaps this interesting property is due to existence of invariant measure for a range of values of parameter of these maps. Hence, to approve this conjecture, it would be interesting to find other measurable one parameter maps, specially higher dimensional maps, which is under investigation.

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